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Initiation of bitopolization of spaces an its geometrical meaning

Satyendra Prasad, Bhoopesh Singh, Dayanand Mishra and Santosh Kumar Singh
Department of Mathematics, Smt. Indira Gandhi
P.G.College., Lalganj, Mirzapur (UP), India

Abstract

In this paper, an attempt has been made to clarify the geometrical meaning of bitopolization of sets as directed by the originator of this topic (J.C.Kelly). As far as we are concern, none of the researchers in the field of bitopological spaces could have touched this standpoint and, therefore, whatever the aspects have been exposed in this paper is quite new. Further, on the basis of geometrical concept, twinned pairwise open sets and pairwise open sets have expressed intuitively. Also, the diagrammatic representation of pairwise open sets have been depicted which elucidates the geometrical concept of pairwise open sets.

Key words- Geometrical meaning of bitopological spaces, development of pairwise open sets of a metric space to a bitopological spaces.

1.01 Introduction

For a long time, in the history of topology, it has been matter confusion that how the concept of open-ness and close-ness of a set of a metric space can be globalized so that every metric space could be seen in that broad sense but the converse is not always true. Of course, the sense of open-ness and close-ness of a set associated with a distance function is not an abstract property, on the contrary, it is quite geometrical, where the nature of a distance function, as here in this case appears, is not merely perceived like the concept of distances used in Euclidian geometry and therefore sometimes this is why, it is said that a topology is nothing but it is an extension of geometry. The two major properties of open sets as well as closed sets of a metric space are given in the following version:

(a) Arbitrary union of open sets is open;
(b) Finite intersection of open sets is open;
(c) Arbitrary intersection of closed sets is closed;
(d) Finite union of closed sets is closed.

Having observed geometrically, the outer out look of an open set of a metric space (that is, the part of a closed set leaving its boundary) one can depict in one's mind that arbitrary collection of open sets can always be viewed as an open set while the arbitrary intersection of open sets can not be open in accordance with our opinion that a singleton set is not an open set. More explicitly, if one considers a closed region of Euclidian space and views through the region having emanating an internal point of it, then the region can be seen clearly like an open set. A singleton set is a closed set in the sense that the point inside the set is itself the limit point of the set. To globalize the concept of open-ness of a set, the first two or the last two of the above four properties of open sets as well as closed sets, associating an empty set and the whole set as open sets, a collection T of subsets of a non-empty set X is constructed which is then called a topology of the set X and the pair (X, T) is called a topological space. Therefore obviously it can be said that the chief aim to define a topology is to distinguish the open sets and closed sets among subsets of X. The there postulates namely

1. The empty set ϕ and the whole set X belongs to $T_;$
2. The arbitrary collection of members of T belongs to $T_;$ and
3. The finite intersection of members of T belongs to T ; given against to define a topology on a non-empty set X does not assure us that each topological space can be a metric space, for the members of T obtained after operating arbitrary unions and finite intersections can be resolved into individuals in various ways. However on imposing some suitable conditions, a topological space may be converted into a metric space and in what follows in that situation the topological space is said to be metrizable. According to the defining way of topology T the members of T are called open sets and their complements are called closed sets while there may be some subsets of X which neither be open nor be closed and therefore the case which appears here at this point suggest to define the relative topologies on X.

Definition (1.01)- Let (X, T) be topological space. Then Y is called a topological sub-space of X if and only if $Y\subset X$ and the sets which are open in Y are precisely the intersection with Y which are open subsets of X. Equivalently, a subset of Y is closed in Y if and only if it is the intersection of Y with a closed subset of X.

Now since $Y\subset X$, therefore here may be some subsets A_i's of X such that $A_i \supset Y$ and A_i's are open set in X but not in Y or neither open nor closed in X. Thus it can be said that a subset of X may not be open as well as closed in X and neither in Y as well.

Example (1.01) Let **X** {a, b, c, d, e, f, g}. Then the number of subsets of X is 2^7.

Now, suppose we have constructed a topology on X by setting as

$$T = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}, X\}$$

The obviously, T is a topology on X, but the subsets of X like : {a, b, c, d}, {a, b, c, e}, {a, b, c, f} etc. are neither open nor closed in X.

In the discussion of topological theory, the indiscrete topology and the discrete topology on X is the weakest and the strongest topologies on X. Also having been introduced an order relation in X, the scope of topology has been extended to ordered topological spaces which is much essential in identifying the metrizibility of a topological space, for a metric preserve an order relation. In this discussion, our aim is not to elaborate the proper meaning of a single topological space but rather to explain the origin of bitopological spaces as well as to visualize its geometrical aspect. Hence, how this achievement can be obtained is nothing but a subtle inspection of different types of metric which is found to be a quite new discussion in the field of bitopological spaces.

Definition (1.02) (pseudo-metric or semi-metric or Quasi -metric)

A pseudo-metric on a set X is a non-negative function d defined for each pair of points of X such that

(i) $d(x, x) = 0$;
(ii) $d(x, y) = d(y, x)$; and
(iii)$d(x, y) \leq (x, z) + d(z, y)$.

Definition (1.03) (Metric) A metric on a set X is a non-negative function d ($d: X \times X \rightarrow R^+$), defined for each pair of point of X, such that

(i) $d(x, x) = 0$;
(ii) $d(x, y) = 0$ iff $y = x$;
(iv) $d(x, y) = d(y, x)$; and
(v) $d(x, y) \leq d(x, z) + d(z, y)$.

However, the first two postulates of metric can be summarized into a single one having writing as $d(x, y) \geq 0$ iff $x \geq y$.

Definition (1.04) (Quasi-pseudo metric)-A Quasi-pseudo metric on a set X is a non- negative function ($d: X \times X \rightarrow R^+$) defined for each pair of point of X, such that

(i) $d(x, x) = 0$;
(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

The followings are examples of common metrics:

1. Let M be any non – empty set and $d: X \times X \rightarrow R^+$ be defined as

$$d(x, y) = 0 \text{ if } x = y$$
$$= 1 \text{ if } x \neq y$$

Then (M, d) is a metric space. The metric d on M is called the discrete metric or trivial metric on M.

2. Let R be the set of all real numbers and let $X \times X \rightarrow R^+$ be defined in the following way:

$$d(x, y) = |x - y|, \forall x, y \in R.$$

Then d(,) is a metric on R called the standard metric on R.

3.Let R^n be the set of all n-tuples $x = (x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n)$ $x_i \in R$ and let $d_1: X^n \times X^n \rightarrow R^+$, $d_2: X^n \times X^n \rightarrow R^+$, $d_3: X^n \times X^n \rightarrow R^+$ be defined respectively on R^n as follows:

(i) $d_1(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

(ii) $d_2(x, y) = \sum_{i=1}^{n} |x_i - y_i|$

(iii) $d_3(x, y) = \max\limits_{1 \leq I \leq n} (|x_i - y_i|)$;

Then $d_1(,)$, $d_2(,)$ and $d_3(,)$ are matrices on R^n. The metric $d_1(,)$ is called the Euclidian metric on R^n.

4. Let C be the set of all complex numbers and let $d: C \times C \to R^+$ be defined in the following way:

$$d(z_1, z_2) = |z_1, z_2|, \text{ for all } z_1, z_2 \in C$$

Then d (,) is a metric on C

5. Let H^∞ (or R^∞) be the set of all real sequences $x = (x_i)$ such that $\sum\limits_{i=1}^{\infty} x_n^2 < \infty$. For any $x = (x_n)$ and $y = (y_n)$ of H^∞, we define $d: H \times H \to R^+$ in the following way:

$$d(x, y) = \left[\sum_{n=1}^{\infty} (x_n - y_n)^2\right]^{\frac{1}{2}} ..$$

Then d (,) is a metric on H^∞ or R^∞. The metric space (R^∞, d) is called the real Hilbert space.

6. Let $B_R[a, b]$ be the set of all bounded real - valued function on the closed bounded interval [a, b] of the real line. For any $f, g \in B_R[a, b]$, we define

$$d(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|.$$ Then d (,) is a metric space on $B_R[a, b]$.

7. Let $C_R[a, b]$ be the set of all continuous function $f: (a, b) \to R$ and let for $f, g \in C_R[a, b]$, d is defined as follows:

$$d(f, g) = \int_a^b |f(x) - g(x)|\, dx,$$

where the integral is taken in Riemann sense. Then d (,) is a metric on $C_R[a, b]$.

8. Let (X, P) be a metric space. Then the metric d (,) defined as is a metric on the set [0,1]

$$d(x, y) = \frac{P(x, y)}{1 + P(x, y)}$$

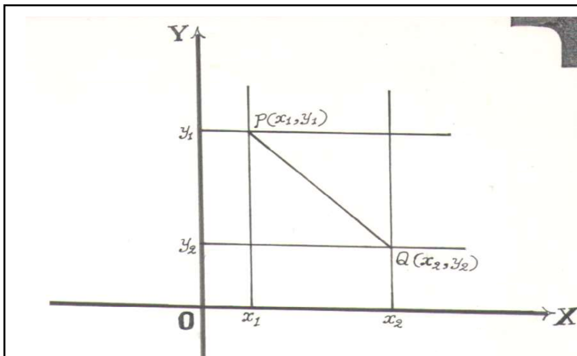

Fig. 1.1

1.02 A minute discussion about metrics

It is quite obvious, having seen the definition of metrics, that all of metrics defined here are non-negative functions which can also be visualized by inserting the postulate $d(x, y) \geq 0$. The originator (Kelly, 1963) of the topic 'Bitopological Space' received on

15th Feb.1962, has enunciated the condition ‘d (x, y) = only if x = y’ by saying this a classical condition. In our discussion we are going to describe its meaning manifestedly.

The metrics defined above satisfy the condition d (x, y) = 0 only if x = y vividly but if one wish to onsider a function like : $d(x, y) = (x_1 - x_2)^3 + (y_1 - y_2)^3$ on $R \times R$, where $P(x_1, x_2)$ and $Q(y_1, y_2)$ are two district points on R^2-plane in first quadrant, then evidently $d(y, x) = (x_2 - x_1)^3 + (y_2 - y_1)^3$, $= [-(x_2 - {}_1)]^3 + [-(y_2 - y_1)]^3 = -(x_1 - x_2)^3 - (y_1 - y_2)^3 = -d(x, y)$. Also, $d(x, y) = 0 \Rightarrow (x_1 - x_2)^3 + (y_1 - y_2)^3 = 0$ which does not imply $x_1 = x_2$ and $y_1 = y_2$ so as to it could be $x = (x_1, y_1) = (x_2, y_2) = y$ i.e. : . = y. As a matter of fact, the expression $(x_1 - x_2)^3 + (y_1 - y_2)^3$ might be zero provided the differences $x_1 - x_2$ and $y_1 - y_2$ are of the same magnitude but opposite in sign although the point x and y will not be found to be coincident as shown in Fig(1.1). If x and y are equal then clearly d (x, y) = 0 without any hesitation. Thus, here in this discussion, we observe that the condition ‘d(x, y) = 0 only if x = y’ splits a metric in two parts out of one of which may be treated as the conjugate of the former. Let us denote these two metrics by p (,) and q (,) so that p(x, y) = q(y, x). Hence the condition ‘d (x, y) = 0 only if x = y’ classifies the metric d(,) on X in two parts provided it will be relaxed from the definition of a metric and if one adjoins the condition p(x, y) = q(x, y) to define quasi and conjugate quasi metrics, then it will generate two quasi metrics space namely (X, p) and (X, q) of which one will be conjugate space to the other. It can be proved that if p(,) is a quasi - psuedo metric, then q(,) is also a quasi - psuedo metric. For, if p(,) is a quasi - psuedo metric, we shall have then

(i) $P(x, x) = 0$ and (ii) $p(x, y) \le p(x, z) + p(z, y)$.

Now, since $p(y, x) = q(y, x)$, so $q(x, x) = p(x, x) = 0$

and $q(x, y) = p(y, z) \le p(y, z) + p(z, x) = q(z, y) + q(x, z) = q(x, z) + q(z, y)$ which shows that q(,) is also a quasi - psuedo metric.

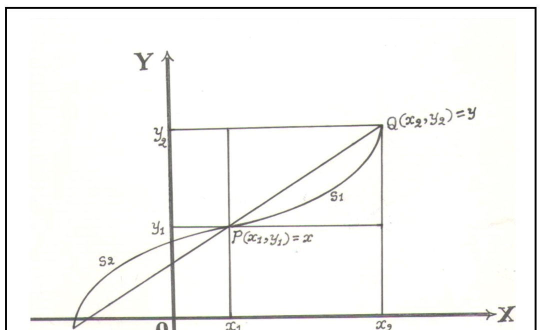

Fig. 1.2

Again, it is a markable fact that the distances shown by a quasi metric and its conjugate are not the same as shown by a common metric. For example, the distance shown by the metric $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is an Euclidian distance PQ, where as the distances shown by the quasi metric $p(x, y) = (x_1 - x_2)^3 + (y_1 - y_2)^3$ and its conjugate $q(x, y) = -p(x, y)$ will be found to be curvilinear distances S1 and S2 as shown in fig. (1.2), where $|x_1 - x_2| = |y_1 - y_2|$. However, there may be some quasi - metrics for which $d(x, y) \neq 0$ in actual sense, yet they may represent quasi - curvilinear distances. The case appears when $|x_1 - x_2| \neq |y_1 - y_2|$. Also, it should be noted that the curvilinear distances represented by a quasi - psuedo metric and its conjugate between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ have the symmetry about the line of shortest distance between them.

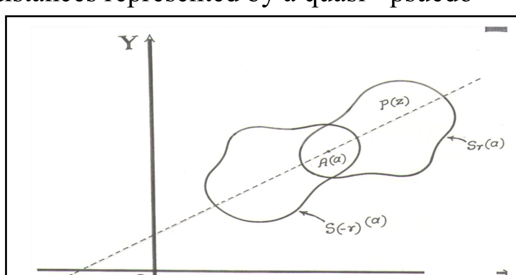

Fig. 1.3

1.03 Geometrical Concept of Bitopolization of sets

As everyone is well acquainted that a metric generates an open sphere which is often symbolized by $S_r(x_0)$ and defined as $S_r(x_0) = \{x: d(x_0, x) < r\}$. Where x_0 is the centre and r is the radius of the sphere. Precisely in the same way a quasi - psuedo metric p(,) or its conjugate q(,) can have the ability to generate open spheres represented as $S_r(a) = \{z : p(a, z) < r\}$ and $S_{(-r)}(a) = \{z : q(a, z) > - r\}$, where a and z are fixed and variable points respectively. In two dimensional space these two spheres have been shown in fig. (1.3) which is symmetrically situated about the central point A (a), where a = (a_0, b_0) is the central point and z = (x, y) a variable point in the plane; the spheres should be viewed having depicted them free from boundary.

Now, if one considers two topologies T_1 and T_2 on X in which T_1- open sets and T_2 - open sets are determined respectively by p(,) and q(,), them the triplet (x T_1, T_2) has been called a bitopological space. Actually, in the study of topology the nature of space has been observed with a view to examine some topological invariant properties as well as the distribution of elements in the space. It is quite evident that if P and Q are T_1-open set and T_2- open set determined by p(,) and q(,) respectively, then $P \cap Q \neq \phi$. If T_1 and T_2 be two arbitrary topologies on X, then one may consider (X, T_1, T_2) also a bitopological space in which $Pi \cap Qi$ may be T_1 - open as well T_2 - open, where $Pi \in T_1$, $Qi \in T_2$ and therefore $Pi \cap Qi$ might be considered as a twinned pairwise open sets of bitopological space. Now-a-days and also prior to sometime, several topologists have been assuming that $Pi \cup Qi$, $Pi \in T_1$, $Qi \in T_1$ is a pairwise open set, but- however, the originator of this topic himself has said nothing about such type of assumption. Actually, a bitopological spaces does not behave like a single topological space on the basis of which $Pi \cup Qi$ should be considered a pair wise open set, yet on the basis of T_1 -open-ness of Pi and T_2 -open-ness of Qi, there is no any harm to assume $Pi \cup Qi$ as a paiwise open set.

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