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Finsler spaces with special (α, β) metric of Douglas type

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Abstract

The notion of Douglas space has been introduced by M. Matsumoto and S. Basco [3], [7] as a generalization of Berwald space from the viewpoint of geodesic equations. It is remarkable that a Finsler space is a Douglas space or is of Douglas type if and only if the Douglas tensor vanishes identically. The present paper is devoted to studying the conditions for some Finsler spaces with (α, β)-metric to be of Douglas type. The theories of Finsler spaces with (α, β)-metric have contributed to the development of Finsler geometry and Berwald spaces with (α, β)-metric have been treated by some authors. Since a Berwald space is a kind of Douglas space, the most noteworthy point of the present paper is to observe that, comparing with the conditions of Berwald space, to what extent the condition of Douglas space relaxes.

Keywords- (α, β)-metric, Douglas space

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1. Preliminaries

Let α(x, y) and β(x, y) be a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and a differentiable one-form $\beta = b_i(x)\, y^i$ in an n-dimensional differentiable manifold M^n. If a Finsler fundamental function in M^n is a function L (α, β) of α and β which is positively homogenous of degree one, then the structure $F^n = (M^n, L(\alpha, \beta))$ is called a Finsler space with (α, β)-metric (Matsumoto, 1992). The space $R^n = (M^n, \alpha)$ is called a Riemannian space associated with F^n (Bacso and Matsumoto,1997). In R^n, we have the Christoffel symbols $\gamma^i_{jk}(x)$ and the covariant differentiation ∇ with respect to $\gamma^i_{jk}(x)$. We shall use the symbols as follows:

$$r_{ij} = \frac{1}{2}(\nabla_j b_i + \nabla_i b_j), \quad s_{ij} = \frac{1}{2}(\nabla_j b_i - \nabla_i b_j), \quad s^i{}_j = a^{ir} s_{rj}, \quad s_j = b_r s^r{}_j.$$

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It is to be noted that $s_{ij} = \frac{1}{2}(\partial_j b_i - \partial_i b_j)$. Throughout the paper the symbols ∂_j and $\dot{\partial}_j$ stand for $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^j}$ respectively. We are concerned with the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$ which is given by

$$2G^i(x,y) = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F), \text{ where } F = L^2/2,\ G^i_j = \dot{\partial}_j G^i \text{ and } G^i_{jk} = \dot{\partial}_k G^i_j.$$

The Finsler space F^n is said to be of Douglas type or called a Douglas space (Bacso and Matsumoto,1997) if $D^{ij} = G^i(x, y)\, y^j - G^j(x, y)\, y^i$ are homogeneous polynomial in y^i of degree three. It has been shown that F^n is of Douglas type if and only if Douglas tensor

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n-1}(G_{ijk}\, y^h + G_{ij}\delta^h_k + G_{jk}\delta^h_i + G_{ki}\delta^h_j),$$

vanishes identically, where $G^h_{ijk} = \dot{\partial}_k G^h_{ij}$ is the hv-curvature tensor of Berwald connection $B\Gamma$, $G_{ij} = G^r_{ijr}$ and $G_{ijk} = \dot{\partial}_k G_{ij}$ [2].

Now we consider the function $G^i(x, y)$ of F^n with (α, β)-metric. According to Kitayama *et al.* 1995; Matsmoto, 1999) they are written in the form

$$2G^i = \gamma^i_{00} + 2B^i, \quad B^i = \frac{E}{\alpha} y^i + \frac{\alpha L_\beta}{L_\alpha} s^i_0 - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} C^* \left(\frac{y^i}{\alpha} - \frac{\alpha}{\beta} b^i\right), \tag{1.1}$$

where we put,

$$E = \frac{\beta L_\beta}{L} C^*, \quad C^* = \frac{\alpha\beta(r_{00} L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha r^2 L_{\alpha\alpha})}, \quad b^i = a^{ij} b_j,\ r^2 = b^2\alpha^2 - \beta^2, \tag{1.2}$$

$b^2 = a^{ij} b_i b_j$ and the subscript α and β in L denote the partial differentiation with respect to α and β respectively. Since $\gamma^i_{00} = \gamma^i_{jk}(x) y^j y^k$ is homogenous polynomial in (y^i) of degree two, we have (Matsmoto, 1998).

Proposition (1.1). A Finsler space F^n with (α, β)-metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are homogeneous polynomials in (y^i) of degree three.

Equation (1.1) gives

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i). \tag{1.3}$$

Here we state the following lemma for the latter frequent use (Hashiguchi *et. al*., 1996).

Lemma. If $\alpha^2 \equiv 0 \pmod \beta$, i.e. $a_{ij}(x)\, y^i y^j$ contains $b_i(x)\, y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x) y^i$ satisfying $\alpha^2 = \beta\, \delta$ and $d_i b^i = 2$.

Through out the paper, we shall say "homogeneous polynomial (s) in (y^i) of degree r" as hp(r) for brevity. Thus γ^i_{00} are hp(2) and if the space is of Douglas type then D^{ij} and B^{ij} are hp (3). Also we have assumed that $\alpha^2 \not\equiv 0 \pmod \beta$, through out the paper.

2. Special (α, β) metric

We shall apply the proposition (1.1) to the (α, β)-metric

$$L = \frac{b_1\alpha^2 + b_2\alpha\beta + b_3\beta^2}{a_1\alpha + a_2\beta}$$

where a's and b's are constants. It is obvious that by homothetic change of α and β this kind of metric may be classified as follows:

$a_1 \neq 0$, $a_2 = 0$, we have the Randers metric $L = \alpha + \beta$ and $L = c_1\alpha + c_2\beta + \dfrac{\beta^2}{\alpha}$. (I)

$a_1 = 0$, $a_2 \neq 0$, we have the Randers metric $L = \alpha + \beta$ and (II)

$$L = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}. \tag{2.2}$$

$a_1 a_2 \neq 0$, we have (III)

$$L = \frac{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}{\alpha + \beta}. \tag{2.3}$$

As for Randers metric we have (Aikou, *et al*., 1990)

Theorem (2.1) A Randers space is of Douglas type, if and only if $s_{ij} = 0$. Then $2G^i = \gamma^i_{00} + \dfrac{r_{00}\, y^i}{L}$

We shall discuss the conditions for F^n with metrics (2.1), (2.2) and (2.3) to be of Douglas type in the following three articles.

3. Finsler space with metric (2.1)

For the metric (2.1), we have

$$L_\alpha = \frac{c_1\alpha^2 - \beta^2}{\alpha^2}, \qquad L_\beta = \frac{c_2\alpha + 2\beta}{\alpha}, \qquad L_{\alpha\alpha} = \frac{2\beta^2}{\alpha^3}.$$

Therefore the value of C* given in (1.2) becomes

$$C^* = \frac{\alpha}{2\beta}\left[\frac{r_{00}(c_1\alpha^2 - \beta^2) - 2s_0(c_2\alpha + 2\beta)\alpha^2}{(c_1 + 2b^2)\alpha^2 - 3\beta^2}\right].$$

Also from (1.3), we have

$$B^{ij} = \frac{\alpha^2(c_2\alpha + 2\beta)}{c_1\alpha^2 - \beta^2}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2}{c_1\alpha^2 - \beta^2}\left[\frac{r_{00}(c_1\alpha^2 - \beta^2) - 2s_0(c_2\alpha + 2\beta)\alpha^2}{(c_1 + 2b^2)\alpha^2 - 3\beta^2}\right](b^i y^j - b^j y^i), \tag{3.1}$$

which may be written as

$$\begin{aligned}(c_1\alpha^2 - \beta^2)[(c_1 + 2b^2)\alpha^2 - 3\beta^2]B^{ij} \\ - \alpha^2[c_2(c_1 + 2b^2)\alpha^3 + 2(c_1 + 2b^2)\alpha^2\beta - 3c_2\alpha\beta^2 - 6\beta^3](s^i_0 y^j - s^j_0 y^i) \\ - \alpha^2[r_{00}(c_1\alpha^2 - \beta^2) - 2s_0\alpha^2(c_2\alpha + 2\beta)](b^i y^j - b^j y^i) = 0.\end{aligned} \tag{3.2}$$

Since α is irrational in (y^i), the equations (3.2) are divided into two equations as follows:

$$\begin{aligned}(c_1\alpha^2 - \beta^2)[(c_1 + 2b^2)\alpha^2 - 3\beta^2]B^{ij} \\ - 2\alpha^2[(c_1 + 2b^2)\alpha^2\beta - 3\beta^3](s^i_0 y^j - s^j_0 y^i)\end{aligned}$$

$$- \alpha^2[r_{00}(c_1\alpha^2 - \beta^2) - 4s_0\alpha^2\beta)]\,(b^i y^j - b^j y^i) = 0, \tag{3.3}$$

$$[(c_1 + 2b^2)\alpha^2 - 3\beta^2](s^i_0 y^j - s^j_0 y^i) - 2s_0\alpha^2\,(b^i y^j - b^j y^i) = 0. \tag{3.4}$$

Equation (3.4) may be written as

$$[(c_1 + 2b^2)\alpha^2 - 3\beta^2](s^i_h\delta^j_k + s^i_k\delta^j_h - s^j_h\delta^i_k - s^j_k\delta^i_h) \tag{3.5}$$

$$- 2\alpha^2[s_h\delta^j_k + s_k\delta^j_h)b^i - (s_h\delta^i_k + s_k\delta^i_h)b^j] = 0.$$

Contracting (3.5) with a^{hk}, we get

$$2[(c_1 + 2b^2)\alpha^2 - 3\beta^2]s^{ij} - \alpha^2(b^i s^j - b^j s^i) = 0. \tag{3.6}$$

Contracting (3.5) with b^h, we get

$$[(c_1 + 2b^2)\alpha^2 - 3\beta^2](b^j s^i_k - s^i\delta^j_k + s^j\delta^i_k - b^i s^j_k) = 0.$$

Contracting it again by j = k, we get

$$n[(c_1+2b^2)\alpha^2 - 3\beta^2]\, s^i = 0. \tag{3.7}$$

Since $\alpha^2 \not\equiv 0(\text{mod } \beta)$ we have $n[(c_1+2b^2)\alpha^2 - 3\beta^2] \neq 0$. Hence from (3.7) we have $s^i = 0$ and consequently (3.6) yields $s^{ij} = 0$ which implies $s_{ij} = 0 = s^i_j$. Putting these values in (3.3), we get

$$[(c_1 + 2b^2)\alpha^2 - 3\beta^2]B^{ij} - \alpha^2 r_{00}\,(b^i y^j - b^j y^i) = 0. \tag{3.8}$$

The term in (3.8) which seemingly does not contain α^2 is $-3\beta^2 B^{ij}$. Hence we must have hp(3) $u^{ij}_{(3)}$, satisfying $-3\beta^2 B^{ij} = \alpha^2 u^{ij}_{(3)}$. Hence we have $B^{ij} = \alpha^2 u^{ij}$, where we have put $u^{ij}_{(3)} = -3\beta^2 u^{ij}$ with hp(1) u^{ij}. Thus (3.8) reduces to

$$[(c_1 + 2b^2)\alpha^2 - 3\beta^2]u^{ij} - r_{00}\,(b^i y^j - b^j y^i) = 0. \tag{3.9}$$

Transvecting (3.9) by $b_i\, y_j$ ($y_j = a_{ij}\, y^i$), we get

$$[(c_1 + 2b^2)\alpha^2 - 3\beta^2]u^{ij}\, b_i\, y_j = r_{00}(b^2\alpha^2 - \beta^2).$$

Now if $[(c_1 + 2b^2)\alpha^2 - 3\beta^2]$ contains $(b^2\alpha^2 - \beta^2)$, then there exists a scalar function $\lambda(x)$ such that $[(c_1 + 2b^2)\alpha^2 - 3\beta^2] = \lambda(x)\,(b^2\alpha^2 - \beta^2)$, which gives $\lambda = 3$ and $b^2 = c_1$ for $\alpha^2 \not\equiv 0(\text{mod } \beta)$. Thus for $b^2 \neq c_1$, $[(c_1 + 2b^2)\alpha^2 - 3\beta^2]$ is a factor of r_{00}. Hence there exists a function h(x) such that $r_{00} = h(x)\,[(c_1 + 2b^2)\alpha^2 - 3\beta^2]$. Therefore we have $r_{ij} = h(x)\,[c_1 + 2b^2)\,a_{ij} - 3b_i\, b_j]$. Since $\nabla_j b_i = r_{ij} + s_{ij}$, and $s_{ij} = 0$, we have

$$\nabla_j b_i = h(x)\,[c_1 + 2b^2)\,a_{ij} - 3b_i\, b_j]. \tag{3.10}$$

Conversely if (3.10) holds, then from (3.1) it follows that $B^{ij} = h(x)\,\alpha^2\,(b^i y^j - b^j y^i)$ which shows that B^{ij} is hp(3). Hence F^n is a Douglas space.

Since we have the

Theorem (3.1) A Finsler space F^n with (α, β)-metric (2.1) for which $c_2 \neq 0$, $b^2 \neq c_1$ and $\alpha^2 \not\equiv 0(\text{mod } \beta)$, is a Douglas space if and only if there exists a scalar function h(x) such that (3.10) holds. In particular if h(x) = 0, then F^n is a Berwald space.

4. Finsler space with metric (2.2)

For the metric (2.2), we have

$$L_\alpha = \frac{c_1\beta + 2\alpha}{\beta}, \qquad L_\beta = \frac{c_2\beta^2 - \alpha^2}{\beta^2}, \qquad L_{\alpha\alpha} = \frac{2}{\beta}.$$

Therefore the value of C* given in (1.2) becomes

$$C^* = \frac{\alpha}{2}\left[\frac{\beta r_{00}(c_1\beta + 2\alpha) - 2\alpha s_0(c_2\beta^2 - \alpha^2)}{c_1\beta^3 + 2b^2\alpha^3}\right]$$

Also from (1.3), we have

$$B^{ij} = \frac{\alpha(c_2\beta^2 - \alpha^2)}{\beta(c_1\beta + 2\alpha)}(s_0^i y^j - s_0^j y^i) + \frac{\alpha^3[\beta r_{00}(c_1\beta + 2\alpha) - 2\alpha s_0(c_2\beta^2 - \alpha^2)]}{\beta(c_1\beta + 2\alpha)(c_1\beta^3 + 2b^2\alpha^3)}(b^i y^j - b^j y^i), \tag{4.1}$$

which may be written as

$$\beta(c_1^2\beta^4 + 2b^2c_1\alpha^3\beta + 2c_1\alpha\beta^3 + 4b^2\alpha^4]B^{ij} - \alpha[c_1c_2\beta^5 + 2c_2b^2\alpha^3\beta^2 - c_1\alpha^2\beta^3 - 2b^2\alpha^5](s_0^i y^j - s_0^j y^i) - \alpha^3[\beta r_{00}(c_1\beta + 2\alpha) - 2\alpha s_0(c_2\beta^2 - \alpha^2)](b^i y^j - b^j y^i) = 0. \tag{4.2}$$

Since α is irrational in (y^i), the equations (4.2) are divided into two equations as follows:

$$\beta(c_1^2\beta^4 + 4b^2\alpha^4]B^{ij} - 2b^2\alpha^4(c_2\beta^2 - \alpha^2)](s_0^i y^j - s_0^j y^i) - 2\alpha^4[\beta r_{00} - s_0(c_2\beta^2 - \alpha^2)](b^i y^j - b^j y^i) = 0. \tag{4.3}$$

And $2(b^2\alpha^2 + \beta^2)B^{ij} - \beta(c_2\beta^2 - \alpha^2)(s_0^i y^j - s_0^j y^i) - r_{00}\alpha^2(b^i y^j - b^j y^i) = 0.$ (4.4)

Only the terms $c_1^2\beta^5 B^{ij}$ of (4.3) seemingly does not contain α^4. Therefore there exists a hp(4) $v_{(4)}^{ij}$ such that $c_1^2\beta^5 B^{ij} = \alpha^4 v_{(4)}^{ij}$. Since $\alpha^2 \not\equiv 0 \pmod{\beta}$, we have $\beta B^{ij} = \alpha^4 v^{ij}(x)$, where v^{ij} is hp(0) i.e. a function of x^i only such that $v_{(4)}^{ij} = c_1^2\beta^4 v^{ij}$. Hence (4.3) is reduced to

$$(c_1^2\beta^4 + 4b^2\alpha^4]v^{ij} - 2b^2(c_2\beta^2 - \alpha^2)](s_0^i y^j - s_0^j y^i) - 2[\beta r_{00} - s_0(c_2\beta^2 - \alpha^2)](b^i y^j - b^j y^i) = 0. \tag{4.5}$$

The terms in (4.5) which seemingly does not contain β is

$$4b^2\alpha^4 v^{ij} + 2b^2\alpha^2)](s_0^i y^j - s_0^j y^i) - 2s_0\alpha^2)(b^i y^j - b^j y^i).$$

Hence we must have hp(1) w^{ij} such that the above is equal to $2\alpha^2\beta w^{ij}$. Hence

$$2b^2\alpha^2 v^{ij} + b^2(s_0^i y^j - s_0^j y^i) - s_0(b^i y^j - b^j y^i) = \beta w^{ij}. \tag{4.6}$$

By putting $w^{ij} = w_k^{ij}(x)y^k$, the above is written as

$$4b^2a_{hk}v^{ij} + b^2[s_h^i\delta_k^j + s_k^i\delta_h^j - s_h^j\delta_k^i - s_k^j\delta_h^i] - [(s_h\delta_k^j + s_k\delta_h^j)b^i - (s_h\delta_k^i + s_k\delta_h^i)b^i] = b_h w_k^{ij} + b_k w_h^{ij}. \tag{4.7}$$

Contracting (4.7) by j = k, we get

$$4b^2 a_{hr}v^{ir} + n(b^2 s_h^i - b^i s_h) = b_h w_r^{ir} + b_r w_h^{ir}. \tag{4.8}$$

Next transvecting (4.7) by $b_j b^h$, we have

$$4b^2 b_k v^{ir} b_r + b^2 (b^2 s^i_k - s^i b_k - b^i s_k) = b^2 b_r w^{ir}_k + b_k b_r w^{ir}_s b^s. \tag{4.9}$$

Transvecting (4.9) by b^k, we get $4b^4 v^{ir} b_r - 2b^4 s^i = 2 b^2 b_r w^{ir}_s b^s$, which gives

$$b_r w^{ir}_s b^s = b^2 (2v^{ir} b_r - s^i), \qquad \text{provided } b^2 \neq 0. \tag{4.10}$$

Substituting the value of $b_r w^{ir}_s b^s$ from (4.10) in (4.9), we obtain

$$b_r w^{ir}_k = 2 b_k v^{ir} b_r + b^2 s^i_k - b^i s_k .$$

Substituting the value of $b_r w^{ir}_k$ from (4.11) in (4.8), we have

$$b_h w^{ir}_r = 4b^2 a_{hr} v^{ir} - 2b_h v^{ir} b_r + (n-1)(b^2 s^i_h - b^i s_h). \tag{4.11}$$

If we put $w^i = \dfrac{1}{n-1}(w^{ir}_r + 2v^{ir} b_r)$, then equation (4.11) gives

$b^2 s^i_h = w^i b_h + b^i s_h - \dfrac{4b^2}{n-1} a_{hr} v^{ir}$ or $b^2 s_{ij} = b_i s_j + w_i b_j - \dfrac{4b^2}{n-1} v_{ij}$, where $w_i = a_{ij} w^j$ and (4.12) $v_{ij} = a_{ih} a_{jk} v^{hk}$. Since s_{ij} and v_{ij} are skew-symmetric tensors, we have $w_i = - s_i$ easily. Hence

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i) - \frac{4}{n-1} v_{ij}. \tag{4.13}$$

To determine r_{ij} we eliminate B^{ij} from equations (4.3) and (4.4) to obtain

$$C(s^i_0 y^j - s^j_0 y^i) + D(b^i y^j - b^j y^i) = 0, \tag{4.14}$$

where $\quad C = (c_2 \beta^2 - \alpha^2)(c_1 \beta^6 - 4b^4 \alpha^6)$

and $\quad D = r_{00} \alpha^2 \beta^3 (c_1^2 \beta^2 - 4\alpha^2) + 4s_0 \alpha^4 (c_2 \beta^2 - \alpha^2)(b^2 \alpha^2 + \beta^2)$. (4.15)

Contracting (4.14) by $b_i y_j$, we get $C s_0 \alpha^2 + D (b^2 \alpha^2 - \beta^2) = 0$, which after substituting the values of C and D gives $s_0(c_2 \beta^2 - \alpha^2)(c_1 \beta^6 - 4 \alpha^2 \beta^4) + r_{00} \beta^3 (c_1^2 \beta^2 - 4\alpha^2)(b^2 \alpha^2 - \beta^2) = 0$.
The terms in (4.15) which seemingly does not contain α^2 is $c_1 c_2 \beta^8 s_0 . - c_1^2 \beta^7 r_{00}$ Hence we must have a function u(x) such that it is equal to $c_1^2 \beta^7 \alpha^2 u(x)$. Therefore $r_{00} = \dfrac{c_2}{c_1} \beta s_0 - \alpha^2 u$. Hence

$$r_{ij} = \frac{c_2}{2c_1}(b_i s_j + b_j s_i) - u \; a_{ij} \tag{4.16}$$

Theorem (4.1) Let F^n be a Douglas space with (α, β)-metric (2.2) for which $b^2 \neq 0$ and $\alpha^2 \not\equiv 0 (\mathrm{mod}\ \beta)$, then there exists a scalar function u(x) and a tensor function $v_{ij}(x)$ such that $\nabla_j b_i$ $(= r_{ij} + s_{ij})$ is given by (4.13) and (4.16).

5. Finsler space with metric (2.3)

For the metric (2.3), we have

$$L_\alpha = \frac{P_2}{(\alpha + \beta)^2}, \qquad L_\beta = \frac{Q_2}{(\alpha + \beta)^2}, \qquad L_{\alpha\alpha} \; \frac{2 c_0 \beta^2}{(\alpha + \beta)^3},$$

where

$$P_2 = c_1\alpha^2 + 2c_1 \alpha\beta + (c_2 - c_3)\beta^2, \; Q_2 = (c_2 - c_1)\alpha^2 + 2c_3 \alpha\beta + c_3 \beta^2, \text{ and } c_0 = c_1 - c_2 + c_3. \tag{5.1}$$

Therefore the value of C* given in (1.2) becomes $C^* = \dfrac{\alpha(\alpha+\beta)(r_{00} P_2 - 2s_0 \alpha Q_2}{2\beta R_3}$,

where

$R_3 = (c_1 + 2c_0 b^2)\,\alpha^3 + 3c_1\,\alpha^2\beta + 3(c_2 - c_3)\,\alpha\,\beta^2 + (c_2 - c_3)\,\beta^3.$ (5.2)

Also from (1.3), we have

$$B^{ij} = \frac{\alpha Q_2}{P_2}(s_0^i y^j - s_0^j y^i) + \frac{c_0 \alpha^3 (r_{00} P_2 - 2s_0 \alpha Q_2)}{P_2 R_3}(b^i y^j - b^j y^i), \qquad (5.3)$$

which may be written as

$$P_2 R_3 B^{ij} - \alpha Q_2 R_3 (s_0^i y^j - s_0^j y^i) - c_0 \alpha^3 (r_{00} P_2 - 2\alpha s_0 Q_2)(b^i y^j - b^j y^i) = 0. \qquad (5.4)$$

From (5.1) and (5.2) we can calculate $P_2 R_3$ and $Q_2 R_3$ as

$P_2 R_3 = p_0\,\alpha^5 + p_1\,\alpha^4\,\beta + p_2\,\alpha^3\,\beta^2 + p_3\,\alpha^2\,\beta^3 + p_4\,\alpha\,\beta^4 + p_5\,\beta^5,$

$Q_2 R_3 = k_0\,\alpha^5 + k_1\,\alpha^4\,\beta + k_2\,\alpha^3\,\beta^2 + k_3\,\alpha^2\,\beta^3 + k_4\,\alpha\,\beta^4 + k_5\,\beta^5,$

where

$$\begin{aligned}
&p_0 = c_1(c_1 + 2\,c_0\,b^2), \quad p_1 = 2p_0 + 3c_1^2, \quad p_2 = (c_2 - c_3)(4c_1 + 2c_0\,b^2) + 6c_1^2, &(5.7)\\
&p_3 = 10\,c_1(c_2 - c_3), \quad p_4 = (c_2 - c_3)[2c_1 + 3(c_2 - c_3)], \quad p_5 = (c_2 - c_3)^2,\\
&k_0 = (c_2 - c_1)\,(c_1 + 2\,c_0\,b^2), \quad k_1 = 3c_1(c_2 - c_1) + 2c_3(c_1 + 2\,c_0\,b^2),\\
&k_2 = 3(c_2 - c_1)\,(c_2 - c_3) + 6c_1\,c_3 + c_3(c_1 + 2\,c_0\,b^2),\\
&k_3 = (c_2 - c_3)\,(c_2 - c_1 + 6c_3) + 3c_1\,c_3, \quad k_4 = 5c_3(c_2 - c_3), \quad k_5 = c_3(c_2 - c_3).
\end{aligned}$$

In view of (5.1), (5.5) and (5.6) the equation (5.4) may be written as

$$(p_0\,\alpha^5 + p_1\,\alpha^4\,\beta + p_2\,\alpha^3\,\beta^2 + p_3\,\alpha^2\,\beta^3 + p_4\,\alpha\,\beta^4 + p_5\,\beta^5)B^{ij} \qquad (5.8)$$

$$- \alpha\,(k_0\,\alpha^5 + k_1\,\alpha^4\,\beta + k_2\,\alpha^3\,\beta^2 + k_3\,\alpha^2\,\beta^3 + k_4\,\alpha\,\beta^4 + k_5\,\beta^5)(s_0^i y^j - s_0^j y^i)$$

$$- c_0\,\alpha^3[r_{00}\{c_1\alpha^2 + 2c_1\,\alpha\beta + (c_2 - c_3)\beta^2\} - 2\alpha\,s_0\{(c_2 - c_1)\alpha^2$$

$$+ 2c_3\,\alpha\beta + c_3\,\beta^2\}](b^i\,y^j - b^j\,y^i) = 0.$$

Since α is irrational in (y^i), the equations (5.8) are divided into two equations as follows:

$$\beta(p_1\,\alpha^4 + p_3\,\alpha^2\,\beta^2 + p_5\,\beta^4)B^{ij} - \alpha^2\,(k_0\,\alpha^4 + k_2\,\alpha^2\,\beta^2 + k_4\,\beta^4)(s_0^i y^j - s_0^j y^i) \qquad (5.9)$$

$$- 2c_0\,\alpha^4\,[c_1\,\beta\,r_{00} - s_0\{(c_2 - c_1)\alpha^2 + c_3\,\beta^2\}](b^i\,y^j - b^j\,y^i) = 0.$$

and $(p_0\,\alpha^4 + p_2\,\alpha^2\,\beta^2 + p_4\,\beta^4)B^{ij} - \beta(k_1\,\alpha^4 + k_3\,\alpha^2\,\beta^2 + k_5\,\beta^4)(s_0^i y^j - s_0^j y^i)$ (5.10)

$$- c_0\,\alpha^2[r_{00}\{c_1\alpha^2 + (c_2 - c_3)\beta^2\} - 4c_3\,s_0\,\alpha^2\,\beta\}](b^i\,y^j - b^j\,y^i) = 0.$$

Only the term $p_5\,\beta^5\,B^{ij}$ of (5.9) seemingly does not contain α^2. Therefore there exists a hp(6) $k_{(6)}^{ij}$ such that it is equal to $\alpha^2\,k_{(6)}^{ij}$. Hence we have $B^{ij} = \alpha^2\,k^{ij}$, where we have put $k_{(6)}^{ij} = p_5\,\beta^5\,k^{ij}$ with hp(1) k^{ij}. Hence (5.9) reduces to

$$\beta(p_1\,\alpha^4 + p_3\,\alpha^2\,\beta^2 + p_5\,\beta^4)k^{ij} - (k_0\,\alpha^4 + k_2\,\alpha^2\,\beta^2 + k_4\,\beta^4)(s_0^i y^j - s_0^j y^i) \qquad (5.11)$$

$$- 2c_0\,\alpha^2\,[c_1\,\beta\,r_{00} - s_0\{(c_2 - c_1)\alpha^2 + c_3\,\beta^2\}](b^i\,y^j - b^j\,y^i) = 0.$$

The terms in (5.11) which seemingly does not contain β are

$$- k_0\,\alpha^4(s_0^i y^j - s_0^j y^i) + 2c_0(c_2 - c_1)\,s_0\,\alpha^4\,(b^i\,y^j - b^j\,y^i).$$

Hence we must have hp(1) m^{ij} such that above is equal to $\alpha^4\beta\,m^{ij}$. Therefore, we have

$$- k_0\,(s_0^i y^j - s_0^j y^i) + 2c_0(c_2 - c_1)\,s_0\,(b^i\,y^j - b^j\,y^i) = \beta\,m^{ij}.$$

By putting $m^{ij} = m_k^{ij}(x)\,y^k$, equation (5.12) may be written as

$$- k_0 [s^i_h \delta^j_k + s^i_k \delta^j_h - s^j_h \delta^i_k - s^j_k \delta^i_h] + 2c_0 (c_2 - c_1) \tag{5.13}$$

$$[(s_h \delta^j_k + s_k \delta^j_h) b^i - (s_h \delta^i_k + s_k \delta^i_h) b^i] = b_h m^{ij}_k + b_k m^{ij}_h$$

Contracting (5.13) by j = k, we get

$$n[- k_0 s^i_h + 2c_0(c_2 - c_1) s_h b^i] = b_h m^{ir}_r + b_r m^{ir}_h . \tag{5.14}$$

Transvecting (5.13) by $b_j b^h$, we obtain

$$- k_0 (b^2 s^i_k - s^i b_k - b^i s_k) = b^2 b_r m^{ir}_k + b_k b_r m^{ir}_s b^s . \tag{5.15}$$

Further transvecting (5.15) by b^k, we get $b_r m^{ir}_s b^s = k_0 s^i$, provided $b^2 \neq 0$. Thus (5.15) gives

$$b^2 b_r m^{ir}_k = k_0 (b^i s_k - b^2 s^i_k) . \tag{5.16}$$

Then (5.14) is rewritten as $b_h m^{ir}_r = -k_0 (n-1) s^i_h + \mu b^i s_h$ (5.17)

where $\mu = 2nc_0 (c_2 - c_1) - \frac{k_0}{b^2}$. If we put $m^i = \mu m^{ir}_r$, then equation (5.17) give

$k_0(n-1) s^i_h = \mu (b^i s_h - m^i b_h)$ or equivalently $s_{ij} = \frac{\mu}{k_0 (n-1)} (b_i s_j - b_j m_i)$.

Since s_{ij} is skew symmetric, we have $m_i = s_i$. Therefore $s_{ij} = \frac{\lambda}{k_0 (n-1)} (b_i s_j - b_j s_i)$. (5.18)

Hence we can state the following

Theorem (5.1) Let F^n be a Douglas space with (α, β)-metric (2.3) for which $b^2 \neq 0$ and $\alpha^2 \not\equiv 0 \pmod{\beta}$, then $(\nabla_j b_i - \nabla_i b_j) = \frac{\mu}{k_0 (n-1)} (b_i s_j - b_j s_i)$, where $\mu = 2nc_0 (c_2 - c_1) - \frac{k_0}{b^2}$, $c_0 = c_1 - c_2 + c_3$. and $k_0 = (c_2 - c_1)(c_1 + 2 c_0 b^2)$.

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