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Rotatory vibration of orthotropic spherical shell

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Abstract

This paper investigates rotatory vibration of orthotropic spherical shell of inner radius a and outer radius b, b>a. Considering the density ρ *of the material of the shell in the form* $\rho = \rho_0 r^n$ *and the elastic constants* $c_{ij} = \mu_{ij} + \mu'_{ij}\frac{\partial}{\partial t} + \mu''_{ij}\frac{\partial^2}{\partial t^2} + \ldots\ldots\ldots + \mu'^{n}_{ij}\frac{\partial^n}{\partial t^n}\ (i, j = 1,2\ldots\ldots n)$ *where* ρ_0 *and* μ'^{s}_{ij} *are constants and n is an integer. The frequency equation of the shell has been derived and graphs have been plotted for discussion of wave so propagated in this case.*

Keywords- Rotatory vibration, Orthotropic material, Spherical shell, Frequency equation.

1. Introduction

Recently, a large number of papers have emerged in the study of vibrational problems in elasticity due to their applications in various branches of science and technology. Mukharjee (1970) discussed the elastic vibration of cylindrical shell of transversely isotropic material. Tranter (1942) investigated the elastic vibration of isotropic cylindrical shell. Narain and Prasad (2006) investigated radial vibration of non-homogeneous composite spherical shell. Narain and Verma (2006) discussed the radial vibration of isotropic cylindrical shell of varying density placed in a magnetic field. Goswami, Sengupta and Chakraborti (2005) discussed radial vibration of composite spherical shell. Narain and Sinha (2006) investigated vibration of visco-elastic spherical shell of variable density. Narain and Sinha (2007) investigated radial vibration of magneto-elastic spherical shell. Sequal to these, the present paper is an attempt to discussed the rotatory vibration of orthotropic spherical shell of inner radius a and outer radius b, b>a. The density ρ and the elastic constants c_{ij} of the material of the shell are assumed to vary as

$$\rho = \rho_0 r^n \text{ and } c_{ij} = \mu_{ij} + \mu'_{ij}\frac{\partial}{\partial t} + \mu''_{ij}\frac{\partial^2}{\partial t^2} + \ldots\ldots\ldots + \mu'^{n}_{ij}\frac{\partial^n}{\partial t^n}\ (i, j = 1,2\ldots\ldots n)$$

respectively where μ'^{s}_{ij} and ρ_0 are constants, n is an integer and r is the radius vector.

2. Fundamental equations and boundary conditions

The stress-strain relations for an orthotropic material in spherical polar co-ordinates (r, θ, ϕ) as given in Love (1944) are :

$$\sigma_{rr} = c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{\phi\phi},\ \sigma_{\theta\theta} = c_{21}e_{rr} + c_{22}e_{\theta\theta} + c_{23}e_{\phi\phi},\ \sigma_{\phi\phi} = c_{31}e_{rr} + c_{32}e_{\theta\theta} + c_{33}e_{\phi\phi} \tag{2.1}$$

$$\sigma_{r\phi} = c_{44}e_{r\phi}, \quad \sigma_{\theta\phi} = c_{55}e_{\theta\phi} \quad \sigma_{r\theta} = c_{66}e_{r\theta}$$

where c_{11}, c_{12} .... etc. are elastic constants and e_{rr}, $e_{\theta\theta}$.......... etc. are strain components. The components of strain in polar co-ordinate (r, θ, ϕ) are,

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r}$$

$$e_{\theta\phi} = \frac{1}{r}\left(\frac{\partial u_\phi}{\partial \theta} - u_\phi \cot\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}, \tag{2.2}$$

$$e_{\phi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \qquad e_{r\phi} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}.$$

We consider

$$c_{ij} = \mu_{ij} + \mu'_{ij}\frac{\partial}{\partial t} + \mu''_{ij}\frac{\partial^2}{\partial t^2} + \dots\dots\dots + \mu'^{n}_{ij}\frac{\partial^n}{\partial t^n}\ (i, j = 1, 2 \dots\dots n)$$

and

$$\rho = \rho_0 r^n \tag{2.3}$$

where μ'^{s}_{ij} and ρ_0 are constants, n is an integer and r is the radius vector.

The boundary condition of the shell is

$$(\sigma_{r\phi})_{r=a} = (\sigma_{r\phi})_{r=b} = 0\ . \tag{2.4}$$

The stress equations of the motion are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta}\cot\theta) = \rho\frac{\partial^2 u_r}{\partial t^2}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r}(\sigma_{\theta\theta} - \sigma_{\phi\phi}\cot\theta + 3\sigma_{r\theta}) = \rho\frac{\partial^2 u_\theta}{\partial t^2}$$

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r}(3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta) = \rho\frac{\partial^2 u_\phi}{\partial t^2} \tag{2.5}$$

For, rotatory vibration of the shell

$$u_r = u_\theta = 0, \quad u_\phi = f(r)\sin\theta e^{i\omega t} \tag{2.6}$$

From the equations (2.1), (2.2) and (2.6) we have,

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = \sigma_{\theta\phi} = \sigma_{r\theta} = 0$$

and

$$\sigma_{r\phi} = (\mu_{44} + i\omega\mu'_{44} - \omega^2\mu''_{44} + \dots)\sin\theta\left(\frac{\partial f}{\partial r} - \frac{f}{r}\right)e^{i\omega t}\ . \tag{2. 7}$$

Using the equations (2.5) and (2.7) we have,

$$\frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}-\frac{2f}{r^2}+\frac{\rho_0 r^n \omega^2 f}{L}=0 \tag{2.8}$$

where,

$$L=\mu_{44}+i\omega\mu_{44}^{'}-\omega^2\mu_{44}^{''}+.... \quad . \tag{2.9}$$

Applying the transformation

$$f(r)=r^{-\frac{1}{2}}F(r) \tag{2.10}$$

the equations (2.8) and (2.10) give

$$\frac{\partial^2 F}{\partial r^2}+\frac{1}{r}\frac{\partial F}{\partial r}-\frac{9}{4}\frac{1}{r^2}F(r)+k^2 r^n F(r)=0 \tag{2.11}$$

where

$$k^2=\frac{\rho_0\omega^2}{L} \quad . \tag{2.12}$$

Using the transformation,

$$Z=\frac{2}{n+2}r^{\frac{n+2}{2}} \tag{2.13}$$

in equation (2.11) we get,

$$\frac{\partial^2 F}{\partial z^2}+\frac{1}{z}\frac{\partial F}{\partial z}+\left(k^2-\frac{\alpha^2}{z^2}\right)F(z)=0 \tag{2.14}$$

where

$$\alpha=\frac{9}{(n+2)^2} \tag{2.15}$$

3. Methods of solution

The solution of the equation (2.13) is given by

$$F(z)=AJ_\alpha(kz)+BY_\alpha(kz) \tag{3.1}$$

where A and B are constants and J_α, Y_α are Bessel's function of first and second kind of order α respectively.

From the equations (2.10), (2.13) and (3.1) we have,

$$f(r)=r^{-\frac{1}{2}}\left\{AJ_\alpha\left(\frac{2k}{n+2}r^{\frac{n+2}{2}}\right)+BY_\alpha\left(\frac{2k}{n+2}r^{\frac{n+2}{2}}\right)\right\} \quad . \tag{3.2}$$

The equation (2.7) and the boundary condition (2.4) give

$$\left(\frac{\partial f}{\partial r}-\frac{f}{r}\right)_{r=a}=0 \quad . \tag{3.3}$$

Further, the equations (3.2) and (3.3) give

$$2ka^{\frac{n+2}{2}}\left[AJ_\alpha^{'}\left(\frac{2k}{n+2}a^{\frac{n+2}{2}}\right)+BY_\alpha^{'}\left(\frac{2k}{n+2}a^{\frac{n+2}{2}}\right)\right]$$

$$-3\left[AJ_\alpha\left(\frac{2k}{n+2}a^{\frac{n+2}{2}}\right)+BY_\alpha\left(\frac{2k}{n+2}a^{\frac{n+2}{2}}\right)\right]=0 \quad . \tag{3.4}$$

Using recurrence formulae

$J'_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$, $Y'_n(x) = \frac{n}{x} Y_n(x) - Y_{n+1}(x)$ in equation (3.4) we have,

$$A\left[\{\alpha(n+2)-3\}J_\alpha\left(\frac{2k}{n+2}a^{\frac{n+2}{2}}\right) - 2k\,a^{\frac{n+2}{2}} J_{\alpha+1}\left(\frac{2k}{n+2}a^{\frac{n+2}{2}}\right)\right]$$
$$+ B\left[\{\alpha(n+2)-3\}Y_\alpha\left(\frac{2k}{n+2}a^{\frac{n+2}{2}}\right) - 2k\,a^{\frac{n+2}{2}} Y_{\alpha+1}\left(\frac{2k}{n+2}a^{\frac{n+2}{2}}\right)\right] = 0 \quad (3.5)$$

Similarly, the equation (3.3) at $r = b$ gives

$$A\left[\{\alpha(n+2)-3\}J_\alpha\left(\frac{2k}{n+2}b^{\frac{n+2}{2}}\right) - 2k\,b^{\frac{n+2}{2}} J_{\alpha+1}\left(\frac{2k}{n+2}b^{\frac{n+2}{2}}\right)\right]$$
$$+ B\left[\{\alpha(n+2)-3\}Y_\alpha\left(\frac{2k}{n+2}b^{\frac{n+2}{2}}\right) - 2k\,b^{\frac{n+2}{2}} Y_{\alpha+1}\left(\frac{2k}{n+2}b^{\frac{n+2}{2}}\right)\right] = 0 \quad (3.6)$$

Eliminating A and B from the equations (3.5) and (3.6) we have,

$$\frac{\{\alpha(n+2)-3\}J_\alpha(\xi) - (n+2)\xi J_{\alpha+1}(\xi)}{\{\alpha(n+2)-3\}J_\alpha(\eta) - (n+2)\eta J_{\alpha+1}(\eta)} = \frac{\{\alpha(n+2)-3\}Y_\alpha(\xi) - (n+2)\xi Y_{\alpha+1}(\xi)}{\{\alpha(n+2)-3\}Y_\alpha(\eta) - (n+2)\eta Y_{\alpha+1}(\eta)} \quad (3.7)$$

where

$$\xi = \frac{2k}{n+2}a^{\frac{n+2}{2}}, \qquad \eta = \frac{2k}{n+2}b^{\frac{n+2}{2}} \quad . \quad (3.8)$$

Expanding J_α, $J_{\alpha+1}$, Y_α, $Y_{\alpha+1}$ and using the result for small x, $J_n(x) = \frac{x^n}{2^n \lfloor n+1}$

in equation (3.7) we get

$$\left(\frac{\xi}{\eta}\right)^{2\alpha}\left[\frac{(n+2)(2\alpha^2-\xi^2)-6\alpha}{(n+2)(2\alpha-2\alpha^2-\xi^2)+6(\alpha-1)}\right] = \left[\frac{(n+2)(2\alpha^2-\eta^2)-6\alpha}{(n+2)(2\alpha-2\alpha^2-\eta^2)+6(\alpha-1)}\right] \quad (3.9)$$

Substituting the value of α, η and ξ from the equations (2.15) and (3.8) in equation (3.9) we have

$$a^{\frac{9}{n+2}}\left[P_1Q_1L^2 + P_2\left(P_1b^{n+2} - Q_1a^{n+2}\right)L - P_2^2(ab)^{n+2}\right]$$
$$= b^{\frac{9}{n+2}}\left[P_1Q_1L^2 + P_2\left(P_1a^{n+2} - Q_1b^{n+2}\right)L - P_2^2(ab)^{n+2}\right] \quad (3.10)$$

where

$$P_1 = 162 - 54(n+2),$$
$$Q_1 = 6(n+2)\{(n+2)^2 + 6\} + 162$$
$$P_2 = 4\rho_0\omega^2(n+2)^2 \quad (3.11)$$

Assuming A'''_{ij}, A^{iv}_{ij}.... etc. to be zero the equations (2.9) and (3.10) give

$$a^{\frac{9}{n+2}}\left[\left\{P_1Q_1\left(\mu_{44}^2 - \omega^2\mu'^2_{44} - 2\omega^2\mu_{44}\mu''_{44} + \omega^4\mu''^2_{44}\right)\right.\right.$$
$$+ 4\rho_0\omega^2(n+2)^2\left(\mu_{44} - \omega^2\mu''_{44}\right)\left(P_1b^{n+2} - Q_1a^{n+2}\right) - 16\rho_0^2\omega^4(n+2)^4(ab)^{n+2}\Big\}$$
$$\left.+ i\left\{2P_1Q_1\left(\mu_{44} - \omega^2\mu''_{44}\right)\mu'_{44}\omega + 4\rho_0\omega^2\mu'_{44}(n+2)^2\left(P_1b^{n+2} - Q_1a^{n+2}\right)\right\}\right]$$
$$= b^{\frac{9}{n+2}}\left[\left\{P_1Q_1\left(\mu_{44}^2 - \omega^2\mu'^2_{44} - 2\omega^2\mu_{44}\mu''_{44} + \omega^4\mu''^2_{44}\right)\right.\right.$$
$$+ 4\rho_0\omega^2(n+2)^2\left(\mu_{44} - \omega^2\mu''_{44}\right)\left(P_1a^{n+2} - Q_1b^{n+2}\right) - 16\rho_0^2\omega^4(n+2)^4(ab)^{n+2}\Big\}$$

$$+i\{2P_1Q_1(\mu_{44}-\omega^2\mu''_{44})\mu'_{44}\omega+4\rho_0\omega^2\mu'_{44}(n+2)^2(P_1a^{n+2}-Q_1b^{n+2})\}\big] \tag{3.12}$$

Comparing real parts on both sides of the equation (3.12) we get,

$$\left[P_1Q_1\mu''^2_{44}+4\rho_0\mu''_{44}\left\{\frac{\left(P_1a^{\frac{9}{n+2}}+Q_1b^{\frac{9}{n+2}}\right)b^{n+2}-\left(P_1b^{\frac{9}{n+2}}+Q_1a^{\frac{9}{n+2}}\right)a^{n+2}}{b^{\frac{9}{n+2}}-a^{\frac{9}{n+2}}}\right\}-16\rho_0^2(n+2)^4(ab)^{n+2}\right]\omega^4$$

$$+\left[\frac{1}{b^{\frac{9}{n+2}}-a^{\frac{9}{n+2}}}\right]\left[P_1Q_1\mu'^2_{44}-4\rho_0\mu_{44}(n+2)^2\left\{\left(P_1a^{\frac{9}{n+2}}+Q_1b^{\frac{9}{n+2}}\right)b^{n+2}-\left(P_1b^{\frac{9}{n+2}}+Q_1a^{\frac{9}{n+2}}\right)a^{n+2}\right\}\right]\omega^2$$

$$+P_1Q_1\mu^2_{44}=0 \tag{3.13}$$

which is the required frequency equation and may be written in the form

$$P\omega^4+Q\omega^2+R=0 \tag{3.14}$$

where

$$P=P_1Q_1\mu''^2_{44}+4\rho_0\mu''_{44}\left\{\frac{\left(P_1a^{\frac{9}{n+2}}+Q_1b^{\frac{9}{n+2}}\right)b^{n+2}-\left(P_1b^{\frac{9}{n+2}}+Q_1a^{\frac{9}{n+2}}\right)a^{n+2}}{b^{\frac{9}{n+2}}-a^{\frac{9}{n+2}}}\right\}\quad-16\rho_0^2(n+2)^4(ab)^{n+2}$$

$$Q=\left[\frac{1}{b^{\frac{9}{n+2}}-a^{\frac{9}{n+2}}}\right]\left[P_1Q_1\mu'^2_{44}-4\rho_0\mu_{44}(n+2)^2\left\{\left(P_1a^{\frac{9}{n+2}}+Q_1b^{\frac{9}{n+2}}\right)b^{n+2}-\left(P_1b^{\frac{9}{n+2}}+Q_1a^{\frac{9}{n+2}}\right)a^{n+2}\right\}\right]$$

$$R=P_1Q_1\mu^2_{44}$$

Numerical calculations

To observe the nature of the frequency of the different materials viz. Quartz, Pyrites (cubic) and Potassium Chloride two cases have been considered. In the first case n=0 while in the second case n=1 with outer radius of the shell being varying in both cases. Graphs have been plotted between the outer radius b and frequency ω.

Table-1 Case-I when $n=0$ and outer radius of the shell is varying

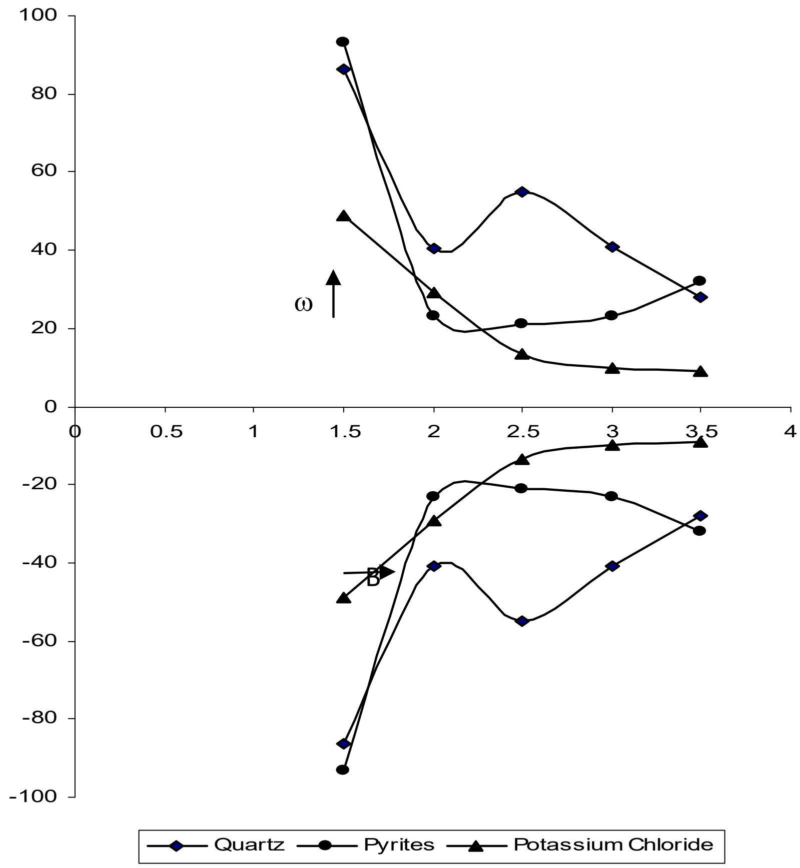
S.No.	Materials	ρ_0	μ_{44}	μ'_{44}	μ''_{44}	a	b	ω
1.	Quartz	2.62	582	-425	2	1	1.5 2.0 2.5 3.0 3.5	$\pm$86.49 $\pm$40.47 $\pm$54.90 $\pm$40.96 $\pm$28.16
2.	Pyrites (cubic)	5.25	1075	-375	4	1	1.5 2.0 2.5 3.0 3.5	$\pm$92.99 $\pm$23.25 $\pm$20.98 $\pm$23.10 $\pm$31.95
3.	Potassium Chloride	1.984	65.50	-350	3	1	1.5 2.0	$\pm$48.90 $\pm$29.22

							2.5 3.0 3.5	± 13.55 ± 09.77 ± 09.07

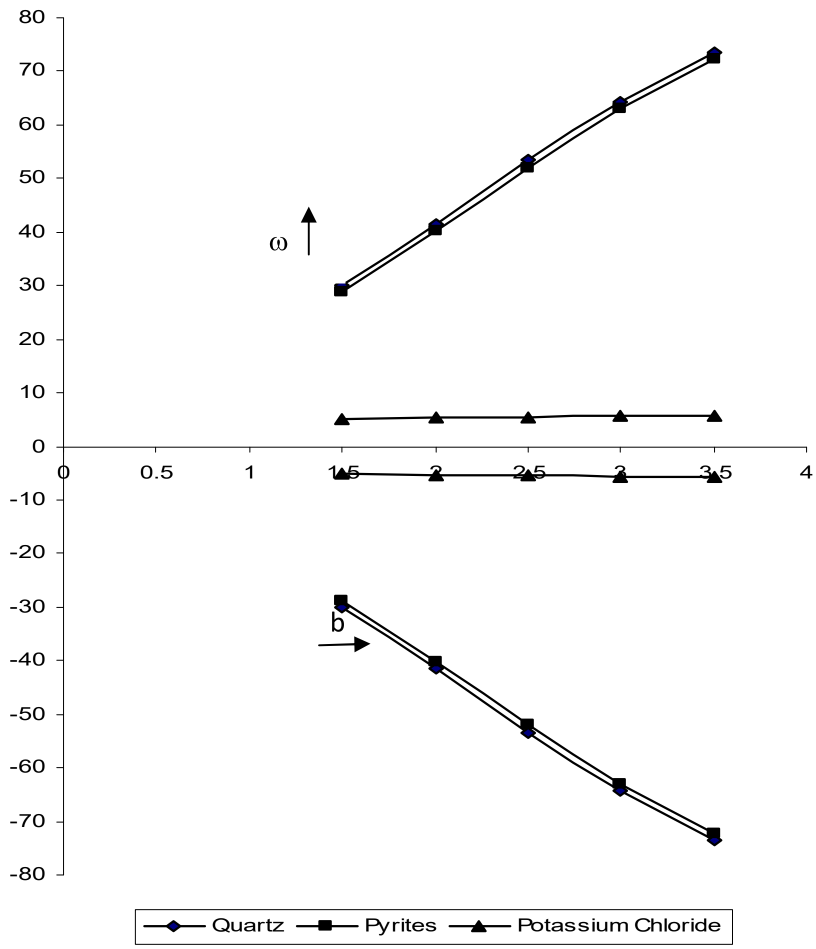
Table-2 Case-II when $n = 1$ and outer radius of the shell is varying

S.No.	Materials	ρ_0	μ_{44}	μ'_{44}	μ''_{44}	a	b	ω
1.	Quartz	2.62	582	-425	2	1	1.5 2.0 2.5 3.0 3.5	± 30.03 ± 41.50 ± 53.37 ± 64.42 ± 73.44
2.	Pyrites (cubic)	5.25	1075	-375	4	1	1.5 2.0 2.5 3.0 3.5	± 28.95 ± 40.15 ± 51.89 ± 63.00 ± 72.25
3.	Potassium Chloride	1.984	65.50	-350	3	1	1.5 2.0 2.5 3.0 3.5	± 05.15 ± 05.41 ± 05.54 ± 05.60 ± 05.63

Graph-1


100
80
60
40
20
0
-20
-40
-60
-80
-100
0
0.5
1
1.5
2
2.5
3
3.5
4
ω
B
Quartz
Pyrites
Potassium Chloride

Graph-2



Conclusions

Case-I

For $n = 0$, from the graph-I we observed that if outer radius of the spherical shell increases frequency of the Quartz decreases rapidly and then increases slowly and further it decreases rapidly, while the frequency of the Potassium Chloride decreases when outer radius increases. In the case of Pyrites (cubic) with the increase of radius of the shell the frequency decreases and then increase gradually.

Case-II

For $n = 1$, from the graph-II we observed that with the increase of outer radius of the shell the frequency of Quartz and Pyrites (cubic) increases rapidly while the frequency of the Potassium Chloride increases very slowly. In both the cases the graphs are mirror image about the line $y = 0$.

References

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